The Comparison of Approximations of Nonlinear Functions Combined with Harmonic Balance Method for Power System Oscillation Frequency Estimation

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*Abstract*—This paper proposes a new approach to estimate the non-constant electro-mechanical oscillation frequencies of a power system by deriving an approximate, analytic expression. The function approximation techniques of Taylor Expansion (TE), Chebyshev Polynomials (CHEB-POL), Padé approximant (PADE) and Continuous Fraction (CONFRAC) representation are combined with the Harmonic Balance Method (HBM) to obtain such an expression. These approaches are illustrated on a Single-Machine-Infinite-Bus (SMIB) system and a 2-Machine System. The TE, CHEB-POL, PADE and CONFRAC are each applied to the swing equation in order to reformulate it into a purely algebraic form. Then, the HBM can be applied in order to derive the approximate, analytical expression describing the oscillation frequencies by considering multiple oscillation components. A numerical integration method is used as a base line when comparing the function approximation techniques. The results demonstrate that CHEB-POL is the superior technique for both SMIB and 2 Machine systems.

***Index Terms***—**Nonlinear Differential Equation, Taylor Expansion, Chebyshev Polynomials, Padé approximant, Continuous Fraction, Harmonic Balance Method**

# Introduction

Understanding the electromechanical oscillations of a power system is critical for maintaining secure and reliable operation. Currently the most accurate technique for studying the electromechanical oscillations is by using numerical integration (NUMINT). For this paper, the NUMINT approach used is the Runge-Kutta (R-K) method. However, NUMINT can be computationally expensive, with extremely long run times for a large system. Therefore, a quicker, more efficient approach for estimating oscillations is needed.

The HBM can be utilized to obtain an explicit expression in the time-domain to describe oscillatory motion. The HBM has been utilized in a variety fields such as aeronautics [10], wireless applications [11] and even analyzing atomic forces [12]. The HBM may also be applied to study frequency oscillations of power systems. Taking advantage of the fundamental relation between the frequency oscillations of a power system and the rotor angle stability, the swing equation can be studied in order to understand frequency oscillations. The HBM is applied to the swing equation whose variable is the rotor angle, $δ\left(t\right)$. The swing equation must be a purely algebraic function. In order to obtain a form of swing equation that is algebraic in nature, a function approximation technique must be applied. Then the HBM can be applied, and the rotor angle expression is derived.

It is important to mention that the methods described in this paper can be derived off-line and can be extended to larger systems. The function approximations can be analyzed and performed much faster than NUMINT, though not as precisely.

The paper is organized as follows. Section II introduces the concepts of each of the four function approximation techniques. Section III introduces the HBM. Section IV describes the applicability to 2-Machine power systems. Section V compares the results each of the four techniques in combination with the HBM compared to the numerical integration approach. Section VI contains the conclusions and suggestions for future work.

# Function Approximation Techniques

The swing equation of an SMIB system, neglecting damping, is

(1)

$$M\ddot{δ}\left(t\right)=P\_{m}-P\_{max}sin⁡(δ\left(t\right))$$

where $M$ is 2 times the generator inertia *H* divided by synchronous speed $ω\_{R}$ $\left(M=\frac{2H}{ω\_{R}}\right)$, $P\_{m}$ is the mechanical power input that represents the operation condition and $P\_{max}$ is the maximum steady-state power output of the generator.

The HBM cannot directly be applied because of the transcendental nature of the sinusoid in the swing equation. Therefore, the swing equation must be reformulated using the function approximation techniques.

For each of the following techniques, terms of order $O(A^{4})$ are negligibly small and ignored.

1. Taylor Expansion

TE is a common function approximation whose terms are calculated from the values of the function’s derivative at an operation point $d\_{o}$. Following the commonly accepted procedure for TE, see reference for further details [8], the SMIB swing equation is rewritten in the form using the third order TE

(2)

$$P\_{m}=M\left(\frac{d^{2}}{dt^{2}}δ(t)\right)+P\_{max}\left(sin\left(d\_{0}\right)+\cos(\left(d\_{0}\right))\left(δ\left(t\right)-d\_{0}\right)-\frac{1}{2}\sin(\left(d\_{0}\right))\left(δ\left(t\right)-d\_{0}\right)^{2}-\frac{1}{6}\cos(\left(d\_{0}\right))\left(δ\left(t\right)-d\_{0}\right)^{3}\right)$$

1. Chebyshev Polynomials

CHEB-POL are recursively defined orthogonal polynomials that are used for function approximation. This paper only studies CHEB-POL of the first-kind. The procedure for defining the polynomials and how to obtain the coefficients of the polynomials is described in detail in reference [4]. The reformulated SMIB swing equation with the 3rd order, first-kind CHEB-POL applied is

(3)

$$P\_{m}=M\left(\frac{d^{2}}{dt^{2}}δ(t)\right)+P\_{max}(0.9999788727δ\left(t\right)-0.1664971411δ\left(t\right)^{3})$$

1. Padé Approximant

PADE expresses functions as a rational function, with both the numerator and the denominator as power series dependent upon a point. The order of the power series for both the numerator and the denominator can be selected. A numerator and denominator with the power of 2 and the operating point $d\_{0}$ is used. The reformulated swing equation is as follows

$P\_{m}=\frac{M\left(\frac{d^{2}}{dt^{2}}δ(t)\right)+2P\_{max}(3f\_{s}^{2}+2f\_{s}f\_{c}^{2}+\left(2f\_{c}^{2}+\frac{5}{2}f\_{s}^{2}f\_{c}\right)f\_{δd}+\left(-\frac{7}{6}f\_{s}f\_{c}^{2}-\frac{5}{4}f\_{s}^{2}\right)f\_{δd}}{6f\_{s}^{2}+4f\_{c}^{2}-f\_{c}f\_{s}f\_{δd}+\left(\frac{1}{2}f\_{s}^{2}+\frac{2}{3}f\_{c}^{2}\right)f\_{δd}}$

(4)

where,

$\left\{\begin{array}{c}f\_{s}=sin⁡(d\_{0})\\f\_{c}=\cos(\left(d\_{0}\right))\\f\_{δd}=(δ\left(t\right)-d\_{0})\end{array}\right.$

(5)

1. Continuous Fraction Representation

CONFRAC typically represents a number as the sum of its integer part and the reciprocal of another number giving the best approximations of irrational numbers [7]. When approximating functions, it is called the Padé approximant. In this paper, CONFRAC refers to a Padé approximant with an operating point $d\_{0}$ equal to zero. The comparison is for determining whether an operating point is necessary for this application.

The SMIB CONFRAC reformulation is

(6)

$$P\_{m}=M\left(\frac{d^{2}}{dt^{2}}δ(t)\right)+\frac{P\_{max}δ(t)}{1+\frac{1}{6}δ\left(t\right)^{2}}$$

1. Approximation Techniques

It is important to note that all of the function approximation techniques approximate the sine and cosine functions very closely on the interval $[\frac{-π}{2},\frac{π}{2}]$. Figures 1 and 2 show the four approximation techniques in comparison to $sin⁡(t)$ and $cos⁡(t)$.



Figure 1. Sine estimation of the approaches



Figure 2. Cosine estimation of the approaches

The function approximation techniques are utilized to estimate the $sin⁡(δ\left(t\right))$ and $cos⁡(δ\left(t\right))$ terms in the swing equation. In order to maintain stable and synchronous operation, the rotor angle, $δ(t)$, should never surpass an angle of $\frac{π}{2}$, or $90°$, for longer than a few cycles. This can be understood by analyzing the power-angle curve, as shown in Figure 3 from EPRI [15].



Figure 3. Power-angle curve

Point “A” in Figure 3 represents a stable operating condition. In practice, the rotor angle is typically around $\frac{π}{4}$ to maintain secure and reliable operation. Point “B” is where the maximum power transfer occurs, but operating at this point can be dangerous to the system. Any slight disturbance can send the rotor angle past the $\frac{π}{2}$, or $90°$, threshold. Operating a system past this threshold for longer than a few cycles will result in loss of synchronism of the system which can be damaging to the system. Therefore, point “C” represents an unstable operation condition and should be avoided.

From the power-angle curve, the rotor angle should maintain an angle less than $\frac{π}{2}$. Therefore, the function approximation techniques should be accurate over the range $[0,\frac{π}{2}]$. By inspection of Figures 1 and 2, all four approximations are reasonably close to the true values for sine and cosine, and therefore the approximation techniques can reliably be utilized to estimate the true values of the transcendental functions.

# Harmonic Balance Method

Once the swing equation is rewritten as a completely algebraic function, the HBM assumption can be applied. The HBM formulation assumes the solution of the SMIB swing equation, $δ(t)$, is the summation of infinite sinusoids.

$δ\left(t\right)=\sum\_{n=0}^{N}A\_{n}cos⁡(nωt+nα)$

(7)

Following the shifting procedure outlined in [1], the calculation of $A\_{0}$ and $α$ is avoided. This paper only considers the first three finite frequency components and assumes there is only one base oscillation frequency *ω*, as shown in equation (8)

(8)

$$δ\left(t\right)=A\_{1}\cos(\left(ωt\right))+A\_{2}\cos(\left(2ωt\right))+A\_{3}cos⁡(3ωt)$$

Then the assumption for $δ\left(t\right)$ is substituted into the reformulated swing equations for each of the function approximation techniques. The equations are then manipulated using common algebraic manipulation and trigonometric identities to reformulate the swing equation to be in the form

(9)

$$C\_{1}\cos(\left(ωt\right))+C\_{2}\cos(\left(2ωt\right))+C\_{3}\cos(\left(3ωt\right))+C\_{o}+f(kωt)$$

$C\_{0}$ is a constant term which can be neglected because of the shift of solution to a standard cosine wave form. $f\left(kωt\right)$ is a sum of higher order harmonic terms whose magnitudes are negligible.

To solve for the magnitudes and oscillation frequency, continue following the HBM procedure by formulating the four equations

$\left\{\begin{array}{c}C\_{1}=0\\C\_{2}=0\\C\_{3}=0\\C\_{1}+C\_{2}+C\_{3}=δ(0)\end{array}\right.$

(10)

Where $C\_{1},C\_{2}$ and $C\_{3}$ are the coefficients of the respective harmonic terms. The bottom equation in (10) is the initial condition equation, which is found using the NUMINT approach. Now there are four equations to solve for four unknown parameters, $A\_{1},A\_{2},A\_{3},$ and ω. Using the 2015 version of the technical computing software Maple, developed by Maplesoft, the function *solve* is used to solve for the parameters.

Since the parameters are of higher order, Maple will return multiple solutions to the equation. The process for selecting the correct oscillation frequency solution is explained by Duan [1].

* Only keep real-value roots and ignore complex roots.
* Only keep roots that satisfy *A*1>>*A*2>*A*3>…>*AN*
* If the *A*1 with a frequency component is larger than 1, then that frequency is not reasonable, because *A*1>1 means *N* oscillation components are not enough to decompose the solution. In the assumed form of solution, at least *N*+1 components are needed to decompose the base frequency component’s magnitude so as to make *A*1 smaller and eventually less than 1.
* When there are multiple *A*1’s<1, select the one closest to the initial value of the shifted solution. (i.e. the actual oscillation magnitude)

The results are demonstrated graphically and in tabular form in Section V.

# 2 machine System

The swing equations for a 2-machine power system are

$\left\{\begin{array}{c}M\_{1}\ddot{δ}\_{1}=P\_{m1}-\left[E\_{1}^{2}G\_{11}+E\_{1}E\_{2}Y\_{12}cos(θ\_{12}-δ\_{1}+δ\_{2})\right]\\M\_{2}\ddot{δ}\_{2}=P\_{m2}-\left[E\_{2}^{2}G\_{22}+E\_{2}E\_{1}Y\_{21}cos(θ\_{21}-δ\_{2}+δ\_{1})\right]\end{array}\right.$

(11)

The formulation of the 2-machine system is slightly more complicated. Now, there are two rotor angles to account for, $δ\_{1}(t)$ and $δ\_{2}(t)$. Neither rotor angle can be represented as a sinusoidal wave form. However, the difference between the two, with $δ\_{1}(t)$ as the reference, does yield a sinusoidal wave form as seen in Figure 4.



Figure 4. Sinusoidal wave form of $δ\_{2}\left(t\right)-δ\_{1}(t)$

Therefore, a new assumption for the HBM must be made, and is expressed

(12)

$$δ\_{2}\left(t\right)-δ\_{1}(t)=Acos\left(ωt\right)+Bcos\left(2ωt\right)+Ccos(3ωt)$$

In order to obtain a formulation to which the HBM assumption can be applied, the swing equations must be manipulated as

(13)

$$M\_{1}\*swing2-M\_{2}\*swing1$$

Terms can be collected, trigonometric identities applied and letting $δ\_{12}\left(t\right)=δ\_{2}\left(t\right)-δ\_{1}(t)$, the resulting equation becomes

(14)

$$0=M\_{1}M\_{2}\ddot{δ\_{12}}\left(t\right)-M\_{1}p\_{m2}+M\_{1}E\_{2}^{2}G\_{22}+M\_{1}E\_{2}E\_{1}Y\_{21}cos\left(θ\_{12}\right)\cos(\left(δ\_{12}\left(t\right)\right))+M\_{1}E\_{2}E\_{1}Y\_{21}\sin(\left(θ\_{12}\right))\sin(\left(δ\_{12}\left(t\right)\right))+M\_{2}p\_{m1}-M\_{2}M\_{1}\ddot{δ\_{1}}-M\_{2}E\_{1}^{2}G\_{11}-M\_{2}E\_{1}E\_{2}Y\_{12}cos\left(θ\_{12}\right)\cos(\left(δ\_{12}\left(t\right)\right))+M\_{2}E\_{1}E\_{2}Y\_{12}\sin(\left(θ\_{12}\right))\sin(\left(δ\_{12}\left(t\right)\right))$$

From this formulation, the four function approximation techniques can be applied to the $\cos(\left(δ\_{12}\left(t\right)\right))$ and $\sin(\left(δ\_{12}\left(t\right)\right))$ terms. Then simply follow the steps as outlined with the SMIB system and apply the HBM to find the magnitude and frequency of each of the oscillation components.

# Result Comparison

The results of comparing TE&HBM, CHEB-POL&HBM, PADE&HBM and CONFRAC&HBM approaches with NUMINT approach for the single-machine and 2-machine system are listed in TABLE I and II, respectively. As shown in Figure 5 and Figure 6, the R-K approach oscillation frequencies are defined as $2π$ times the reciprocal of the period between the first two peaks. In order to change the operating conditions of the system, the fault duration was altered from 1 cycle to 29 cycles. 29 cycles is the marginal stability case.



Figure 5. Single-machine system variation of oscillation frequencies under different operating conditions



Figure 6. 2-machine system variation of oscillation frequencies under different operating conditions

The estimated oscillation frequencies by the TE&HBM, CHEB-POL&HBM, PADE&HBM and CONFRAC&HBM approaches under different operating conditions are listed and compared in TABLE I and TABLE II.

TABLE I. SMIB SYSTEM OSCILLATION FREQUENCIES ESTIMATION COMPARISON

|  |  |  |  |
| --- | --- | --- | --- |
| **Fault Duration (cycles)** | **Estimated Oscillation frequencies (rad/s)** | **Error****(rad/s)** | **R-K****(rad/s)** |
| 1 | TE | 11.733 | 0.309 | 11.424 |
| CHEB-POL | 11.710 | 0.286 |
| PADE | 12.117 | 0.693 |
| CONFRAC | 11.712 | 0.298 |
| 5 | TE | 11.725 | 0.638 | 11.087 |
| CHEB-POL | 11.700 | 0.613 |
| PADE | 12.113 | 1.025 |
| CONFRAC | 11.702 | 0.615 |
| 9 | TE | 11.631 | 0.544 | 11.087 |
| CHEB-POL | 11.598 | 0.511 |
| PADE | 12.028 | 0.941 |
| CONFRAC | 11.604 | 0.527 |
| 13 | TE | 11.388 | 0.618 | 10.770 |
| CHEB-POL | 11.325 | 0.555 |
| PADE | 11.828 | 1.058 |
| CONFRAC | 11.358 | 0.588 |
| 17 | TE | 11.014 | 0.826 | 10.189 |
| CHEB-POL | 10.884 | 0.695 |
| PADE | 11.552 | 1.363 |
| CONFRAC | 10.999 | 0.811 |
| 21 | TE | 10.586 | 0.919 | 9.666 |
| CHEB-POL | 10.331 | 0.665 |
| SSA | 10.992 | 1.326 |
| PADE | 11.233 | 1.567 |
| CONFRAC | 10.612 | 0.946 |
| 25 | TE | 10.166 | 1.588 | 8.568 |
| CHEB-POL | 9.710 | 1.142 |
| PADE | 11.051 | 2.483 |
| CONFRAC | 10.234 | 1.666 |
| 29 | TE | 9.613 | 3.001 | 6.612 |
| CHEB-POL | 8.599 | 1.987 |
| PADE | 10.887 | 4.275 |
| CONFRAC | 9.769 | 3.147 |

TABLE II. 2 MACHINE SYSTEM OSCILLATION FREQUENCIES ESTIMATION COMPARISON

|  |  |  |  |
| --- | --- | --- | --- |
| **Fault Duration (cycles)** | **Estimated Oscillation frequencies (rad/s)**  | **Error** **(rad/s)** | **R-K****(rad/s)** |
| 1 | TE | 5.411 | 0.025 | 5.385 |
| CHEB-POL | 5.404 | 0.019 |
| PADE | 5.550 | 0.165 |
| CONFRAC | 5.411 | 0.026 |
| 5 | TE | 5.399 | 0.012 | 5.387 |
| CHEB-POL | 5.394 | 0.006 |
| PADE | 5.542 | 0.155 |
| CONFRAC | 5.402 | 0.015 |
| 9 | TE | 5.374 | 0.056 | 5.311 |
| CHEB-POL | 5.363 | 0.052 |
| PADE | 5.518 | 0.234 |
| CONFRAC | 5.375 | 0.083 |
| 13 | TE | 5.302 | 0.067 | 5.235 |
| CHEB-POL | 5.300 | 0.066 |
| PADE | 5.469 | 0.235 |
| CONFRAC | 5.318 | 0.083 |
| 17 | TE | 5.180 | 0.084 | 5.196 |
| CHEB-POL | 5.184 | 0.089 |
| PADE | 5.379 | 0.116 |
| CONFRAC | 5.211 | 0.262 |
| 21 | TE | 4.971 | 0.075 | 4.897 |
| CHEB-POL | 4.984 | 0.087 |
| PADE | 5.227 | 0.330 |
| CONFRAC | 5.029 | 0.132 |
| 25 | TE | 4.629 | 0.142 | 4.487 |
| CHEB-POL | 4.646 | 0.159 |
| PADE | 4.985 | 0.498 |
| CONFRAC | 4.732 | 0.245 |
| 29 | TE | 4.065 | 0.669 | 3.396 |
| CHEB-POL | 4.013 | 0.617 |
| PADE | 4.542 | 1.146 |
| CONFRAC | 4.123 | 0.817 |

Figure 7 graphically shows how the oscillation frequencies are estimated by five approaches with the variation of the operating conditions for a single-machine system.

Figure 7. Single-Machine oscillation frequencies comparison between different function approximation approaches

For 2-machine system, Figure 8 shows the estimation of the oscillation frequencies by the five approaches for the 2-machine system

Figure 8. 2-Machine oscillation frequencies comparison between different function approximation approaches

As clearly seen from the results, CHEB-POL is the best function approximation technique to combine with the HBM to estimate power system oscillation frequencies for single- and 2-machine systems. Another advantage of the CHEB-POL technique is that it is not dependent on an operating point $δ\_{0}$ like the TE and PADE methods. CONFRAC also has this advantage and performs better than the PADE. This is possibly because the procedure for selecting a proper operating point, $δ\_{0}$, for simulations is not well-defined. Further research into how to select the $δ\_{0}$ operating point is desirable. In the field, the practical significance of not requiring a $δ\_{0}$ is that the system operator will not have to obtain the operating point, which can save time and money when attempting to find the oscillation frequencies of a system.

# Conclusion

This paper proposes new methods for obtaining approximate, analytic expressions of oscillation frequencies of power systems. The CHEB-POL&HBM method was proven to be superior for the single-machine and 2-machine power systems. A significant advantage of the CHEB-POL&HBM frequency estimation technique is that it is not dependent on a specific operation point, which can be difficult to obtain for system operators. The practical significance of this project is that system operators can use the method offline to deduce the analytic frequency expressions and then use the expressions to run contingency plans. If the system operator finds that the frequency of a system is unacceptable, then the utility will be able to enact anticipatory preventative measures to avoid outages. Recommendations for future study for this topic include testing the scalability of the Chebyshev Polynomials combined with the HBM to more complex power systems. Research involving Chebyshev polynomials that use higher orders, such as 5th order, may potentially provide more accurate results while still maintaining ease of calculations. Also testing Chebyshev Polynomials of the first-, second-, third-, and fourth-kind is necessary to discover which kind of Chebyshev Polynomial yields the best results.

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